4

RELATIONS AND IT'S PROPERTIES

Unit Structure

4.0 OBJECTIVES:

After going through this chapter you will be able to :

- Understand the concept and definition of product and partition of a set.
- Understand the different representation of a relation (set theoretical, pictorial and matrix representation).
- Understand the definition of a path in a relation and able to find paths of different length.
- Understand the different properties of binary relation.

4.1 INTRODUCTION:

In day today life we deal with relationships such as an employee and employee number, element and set, a person and his telephone number etc. In mathematics it's looked in more abstract sense such as division of integers, order property of Real numbers and so on. In computer science, a computer programme and variable, computer language and valid statement and so on Relations are useful in computer databases, networking etc.

4.2 PRODUCT SETS and PARTITIONS:

4.2.1 Product sets

Definition:

Let A and B be two non empty sets. The product set or Cartesian product of A and B, (denoted by $A \times B$) is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Thus, $A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$

[**Note:** an order pair (a, b) is the ordered collection that has 'a' and 'b' in prescribed order, 'a' in first position and 'b' in second position.]

Examples:

(1) Let $A = \{1,\ 2,\ 3\}$ and $B = \{x, y\}$

then $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

Similarly, $B \times A = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

(2) Let A be the set of all 2 divisions in X^{th} class in some school and B be the set of all 3 courses available.

i.e. $A = \{X, Y\}$, $B = \{C^{++},\ \text{Java},\ \text{VB}\}$

then

$A \times B = \{(X,\ C^{++}),\ (X,\ \text{Java}),\ (X,\ \text{VB})\,(Y,\ C^{++}),\ (Y,\ \text{Java}),\ (Y, \text{VB})\}$

so there are total 6 categories possible.

Remark:

(1) $A \times B$ and $B \times A$ may or mayn't be equal.

(2) If A and B are finite sets then $|A \times B| = |A|.|B| = |B \times A|$

(3) An ordered pair (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$.

(4) The idea of Cartesian product of two sets can be extended to 'n' number of sets $A_1, A_2, ..., A_n$, (it's denoted by $A_1 \times A_2 \times ... \times A_n$) and it's defined as, $A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) / a_1 \in A_1, a_2 \in A_2 ... \&\ a_n \in A_n\}$

[$(a_1, a_2, ..., a_n)$ is an ordered 'n' -tuple.]

4.2.2 Partitions

Definition:

A partition of a nonempty set A is a collection $p = \{A_1, A_2, ..., A_n\}$ of nonempty subsets of A such that

(1) $A_1 \cup A_2 \cup A_3 \cup ... \cup A_n = A$

(2) $A_i \cap A_j = \emptyset \ (1 \le i < j \le n)$

$A_1, A_2, A_3, ..., A_n$ are called as blocks or cells of the partition.

Example:

(1) $A = \{1, 2, 3, 4, 5\}$

(a) Let $A_1 = \{1\}$, $A_2 = \{2, 3\}$ and $A_3 = \{4, 5\}$

Then we have, $A_1 \cup A_2 \cup A_3 = A$

$\& A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, $A_2 \cap A_3 = \emptyset$

$\therefore P = \{A_1, A_2, A_3\}$ is a partition of 'A'

(b) Let $A_1 = \{1, 2\}$, $A_2 = \{3\}$, $A_3 = \{4\}$ and $A_4 = \{5\}$

then $P = \{A_1, A_2, A_3, A_4\}$ is a partition of A.

(c) Let $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{4, 5\}$

then $P = \{A_1, A_2, A_3\}$ is not a partition of A

$\therefore A_1 \cap A_2 \ne \emptyset$

(d) Let $A = \mathbb{Z} =$ set of all 'integers'

E = set of all even integers and

O = set of all odd integers

We have, $E \cap O = \varnothing$ and $E \cup O = A$

$\therefore P = \{E,\ O\}$ is a partition of A.

Check your progress

1. List all partition of $A = \{a,\ b,\ d\}$.

2. Let $A = \{a, b, c, d, e, f, g, h\}$ and $A_1 = \{a\}, A_2 = \{b, c\}, A_3 = \{d, e, f\}$, $A_4 = \{g, h\}$ and $A_5 = \{f, g\}, A_6 = \{a, b, c\}$

   Which of the following are partition of A.

   (a) $\{A_1,\ A_2,\ A_3\}$ (b) $\{A_1,\ A_2,\ A_3,\ A_4\}$ (c) $\{A_3,\ A_4,\ A_5,\ A_6\}$

   (d) $\{A_3,\ A_6,\ A_4\}$

3. $A \times B = B \times A$ if (a) A is finite (b) A=B

   (c) B is finite

4. If $A = \{x,\ y,\ z\}$, $B = \{1,\ 2,\ 3\}$ and $C = \{a,\ b\}$ write down the set $A \times B \times C$.

4.3 RELATIONS AND DIAGRAPHS:

4.3.1 Definition and examples of Relation

Definition:

Let A and B be two non empty sets A relation R from A to B is a subset of $A \times B$.

If $(x,\ y) \in R$ then we write xRy and If $(x,\ y) \notin R$ then we write $x \not R y$.

Examples:

(1) Let $A = \{1,\ 2,\ 3\}$ & $B = \{x,\ y,\ z\}$ then $R = \{(1,\ x), (1,\ y), (2,\ z)\}$ is relation from A to B.

Note: If A=B then instead of saying a relation from A to B we will say a relation on A.

(2) Let $A = B = \{1,\ 2,\ 3,\ 4\}$, Let R be a relation on A defined as xRy iff $x > y$.

$\therefore R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

(3) Let $A = B = \mathbb{R} =$ The set of Real numbers.

Let R be a relation on A such that

xRy iff $x^2 + y^2 = 25$

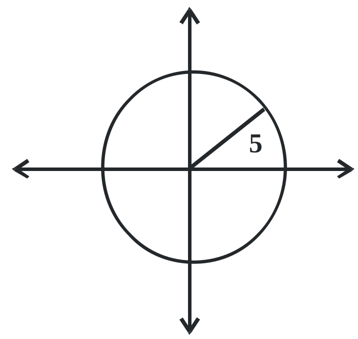


Fig. 4.1

$\therefore R = \left\{(x,\ y) / x^2 + y^2 = 25\right\}$

= The set of all points on the circle centre at origin with radius ‘5’

We can see, $(3,4) \in R \quad (\because 3^2 + 4^2 = 25)$

but $(3, 3) \notin R \quad (\because 3^2 + 3^2 = 18 \neq 25)$

(4) Let $A = \mathbb{N}$, Let ‘R’ be a relation on ‘A’ defined as xRy iff ‘x’ divides ‘y’. $R = \{(1, 2), (2, 4), (5, 10), (2, 6), ...\}$

We have 1 R 2 but $2 \not R\ 1$.

4.3.2 Sets related to a relation

Let ‘R’ be relation from A to B.

Two important sets related to R are the Domain of ‘R’ [denoted by Dom (R)] and The Range of R [denote by Ran (R)].

We have, Dom (R)= $\{x / (x, y) \in R\} \subseteq A$ i.e. Dom (R) is a subset of ‘A’ containing first element of the pair (x, y) which belongs to ‘R’. Similarly, Ran (R)= $\{y / (x, y) \in R\} \subseteq B$.

For Example:

(1) Let $A = \{1,\ 2,\ 3\}$ and $B = \{x, y\}$ $R = \{(1,\ x), (3,\ y)\}$

Dom (R)= $\{1, 3\} \subseteq A$ and Ran (R)= $\{x, y\} = B$

(2) Let A and B be only two sets and R = $A \times B$.

Then Dom (R) = A and Ran (R) =B

Check your progress

1. Write down the elements of R form $A = \{0, 1, 2, 3\}$ to $B = \{1, 2, 3\}$, defined as $(a, b) \in R$ iff

   (a) a = b (b) a + b is an even number

   (c) a + b is a multiple of '3' (d) $a \geq b$

2. Find the domain and Range of the relations defined in Q.1.

4.3.3 The matrix of a Relation

A relation between two finite sets can be represented by a Boolean matrix (a matrix which is having entries as '0' or '1')

Let 'R' be a relation from $A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$. (Here the elements of A and B are listed in a particular order). Then relation 'R' can be represented by the *m* x *n* matrix $M_R = [mij]_{m\times n}$, which is defined as,

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called as the matrix of a Relation 'R'

Examples:

(1) Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$ and

$R = \{(1, x), (2, x), (3, y), (1, y), (3, x)\}$

$$\therefore M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

(2) Let $A = B = \{1, 2, 3, 4\}$

Let 'R' be a relation on 'A' defined as xRy iff $x \leq y$.

$\therefore R = \{(1,1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

$$\therefore M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

Note: Converse process is also possible i.e. given a matrix with enteries '0' or '1' we can write 'R' related to that matrix.

4.3.4 The Diagraph of a Relation

Just we saw that a relation on finite set 'A' can be represented by a binary matrix. Similarly there is another way of representing a relation using a pictorial representation. Pictorial representation of 'R' is as follows, Draw a small circle for each element of A and label the circle with the corresponding elements of A, (these circles are called as vertices) draw an arrow from vertex a_i to a_j iff $a_i \, R \, a_j$. (these arrows are called as edges)

The resulting pictorial representation is called as Directed graph or diagraph of 'R'.

For example:

(1) Let $A = B = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1, 2), (2, 3), (3, 4), (2, 4), (3,1)\}$

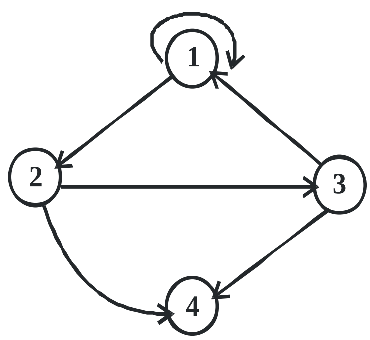


Fig. 4.2

Note:

(1) An edge of the form (a, a) is represented using an arc from the vertex 'a' back to it self. Such an edge is called a loop.

(2) Conversely diagraph can be used to find underlying relation represented by it.

(3) There are two important definitions arising from the diagraph.

(i) In-degree of a vertex = no. of arrows coming towards that vertex and

(ii) Out-degree of a vertex = no. of arrows going away from that vertex

For example (1) for below diagraph

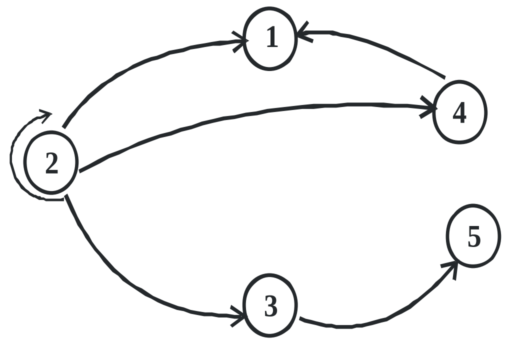


Fig. 4.3

	1	2	3	4	5
In-degree	2	1	1	1	1
Out-degree	0	4	1	1	0

Check your progress

1. Write down the matrix M_R and draw the diagraph for following relations.

   (a) $A = B = \{1, 2, 3, 4\}$, R is such that xRy iff $x \mid y$.

   (b) $A = B = \{1, 2, 3, 4\}$, 'R' is such that xRy iff $x + y \leq 5$

   (c) $A = B = \{1, 2, 4, 6\}$, 'R' is such that xRy iff $x + y$ is a multiple of '2'

2. Write down In-degree and out-degree for each of the vertices in example (1).

3. Let $A = \mathbb{R}$, Give a description of the relation 'R' specified by the shaded region shown below

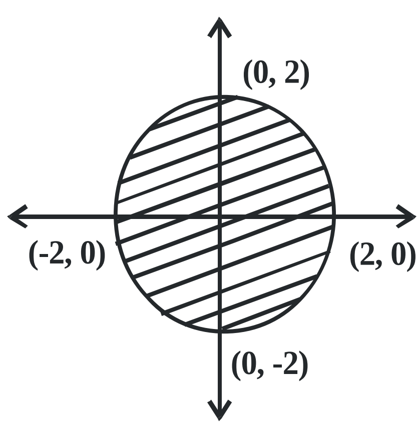


Fig. 4.4

4.4 PATHS IN RELATIONS and DIAGRAPHS:

Let A be a given set and let ‘R’ be a relation on ‘A’. A path of length m in R from ‘a’ to ‘b’ is a finite sequence. $\pi: a = x_0,\ x_1,\ x_2,\ x_3, ..., x_{m-1},\ x_m = b,$ starting from ‘a’ and ending to ‘b’ such that $aRx_1,\ x_1\,Rx_2,\ x_2\,Rx_3, ..., x_{m-1}\,R\,b.$

Note: Length of a path is nothing but the number of arrows involved in a path.

For example:

(1) Consider the following diagraph,

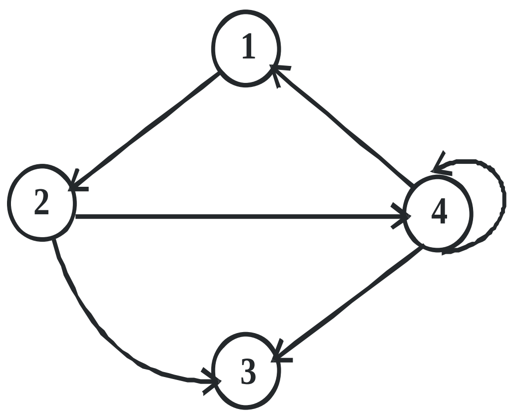


Fig. 4.5

$\pi_1 : 1,\ 2, 4$ is a path of length ‘2’

$\pi_2 : 1, 2, 4, 1$ is a path of length ‘3’

$\pi_3 : 4, 4, 3$ is a path of length ‘2’

$\pi_4 : 4, 4$ is a path of length ‘1’

Note: a path like $\pi_2 \,\&\, \pi_4$ are called as cycles, a cycle is a path which is having same starting and ending vertex.

4.4.1 Paths in a relation ‘R’ can be used to define new relations.

From above example, It can be seen that paths of length 1 can be identified with the elements of R and vice versa. So ‘R’ can be replaced by R^1 where ‘1’ stands for set of all order pairs (x, y) for which there exist a path of length 1 from x to y. on similar lines now we can define $R^2,\ R^3, ..., R^n\ (n \in \mathbb{N})$

$$R^2 = \{(x,\ y)/\text{if a path of length '2' from 'x' to 'y'}\}.$$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$$R^n = \{(x,\ y)/\text{if a path of length 'n' from 'x' to 'y'}\}.$$

$\vdots \qquad\qquad\qquad\qquad \vdots$

Now we may define R^{∞} as,

$$R^{\infty} = \{(x,\ y)/\text{if some path from 'x' to 'y'}\}.$$

i.e. $\boxed{R^{\infty} = \bigcup_{i=1}^{\infty} R^i}$

For example:

(1) Consider the following diagraph.

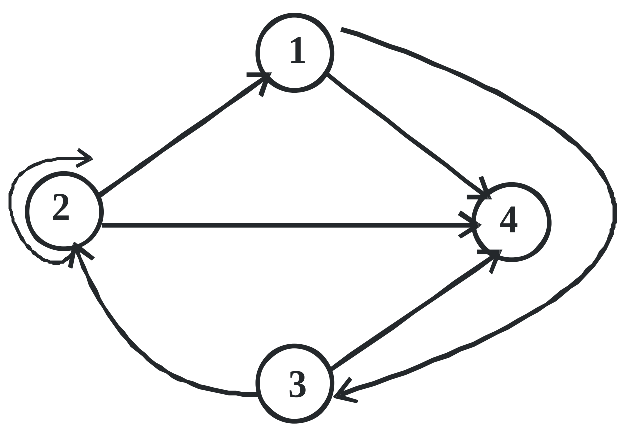


Fig. 4.6

So, $R^1 = \{(2, 2), (2, 1), (2, 4), (3, 2), (3, 4), (1, 4), (1, 3)\}$

Now for R^2, we have to find all Paths of length ‘2’

$\pi_1 : 2, 2, 2 \qquad \pi_2 : 2, 2, 1 \qquad \pi_3 : 2, 2, 4$

$\pi_4 : 3, 2, 2 \qquad \pi_5 : 3, 2, 4 \qquad \pi_6 : 3, 2, 1$

$\pi_7 : 1, 3, 2 \qquad \pi_8 : 1, 3, 4 \qquad \pi_9 : 2, 1, 3$

i.e. $R^2 = \{(2, 2,), (2, 1), (2, 4), (3, 2), (3, 4), (3, 1), (2, 3), (1, 2), (1, 4)\}$

similarly we can find R^3, R^4 and so on.

(2) Consider the diagraph,

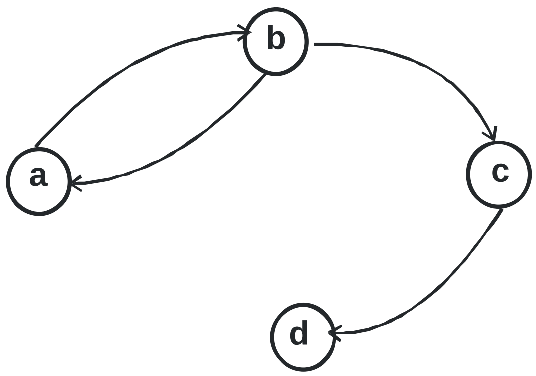


Fig. 4.7

Then, $R^1 = \{(a,b),(b,a),(b,c),(c,d)\}$

$R^2 = \{(a,a),(a,c),(b,b),(b,d)\}$

$R^3 = \{(a,b),(a,d),(b,a),(b,c)\}$

$\vdots \qquad\qquad \vdots$

$R^\infty = \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,d)\}$

Note: Since 'n' is finite the process of finding $R^1, R^2, R^3, \ldots$ will stop after some finite 'n'.

In fact we can prove that, $R^\infty = R^1 \cup R^2 \cup R^3 \cup \ldots\ldots \cup R^n$, where 'n' is number of elements in the given set 'A'.

4.4.2 Matrix version

If $|R|$ is large, it would be tedious to compute R^∞, or even R^2, R^3 etc. from the set representation of R so we have following matrix version of above concepts.

First we will see some different operations defined on Boolean matrices.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two m x n Boolean matrices.

(1) we define $A \vee B = C = [C_{ij}]$, the join of A and B by,

$$C_{ij} = \begin{cases} 1 & \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if } a_{ij} \text{ \& } b_{ij} \text{ both are } 0 \end{cases}$$

(2) We define $A \wedge B = E = [e_{ij}]$, the meet of A and B

$$e_{ij} = \begin{cases} 1 & \text{if } a_{ij} = 1 \,\&\, b_{ij} = 1 \\ 0 & \text{if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

(3) Let $A = \left[a_{ij}\right]_{m\times p}$ and $B = \left[b_{ij}\right]_{p\times n}$ be two Boolean matrices.

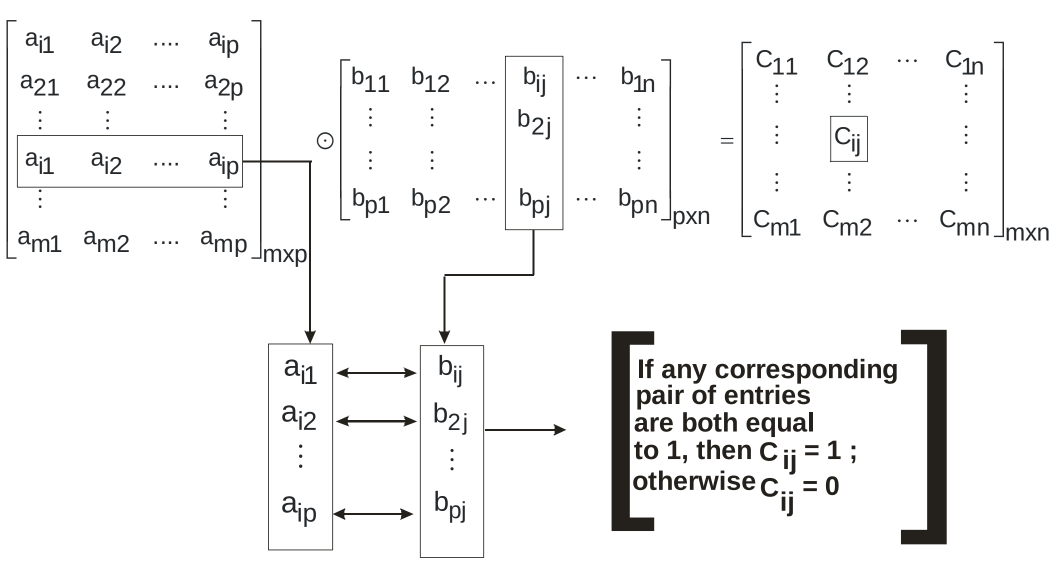
Then Boolean product of A and B, (denoted by $A \odot B$) is the $m \times n$ Boolean matrix $C = [C_{ij}]$ defined by

$$C_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some k, } 1 \le k \le p \\ 0 & \text{otherwise} \end{cases}$$

for eg: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$ & $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

$$\therefore A \odot B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

Note:



We can prove that $\boxed{M_{R^2} = M_R \odot M_R}$ i.e. matrix related with R^2 is nothing but the Boolean product of M_R with M_R similarly. $\boxed{M_{R^n} = M_R \odot M_R \odot .... \odot M_R}$ ('n' times)

For example

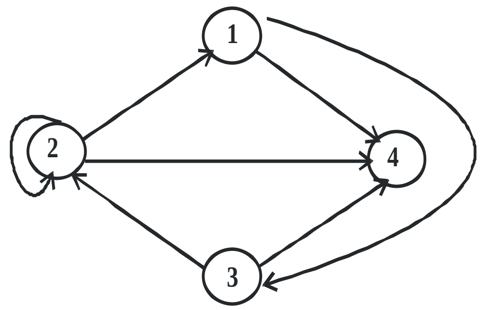


Fig. 4.8

as we saw, $R^2 = \{(2,1), (2,2), (2,4), (3,2), (3,4), (3,1), (1,2), (2,3), (1,4)\}$

$$\therefore M_{R^2} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \quad \text{and} \quad M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

Now let's compute, $M_R \odot M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{which is nothing but } M_{R^2}$$

i.e. $M_{R^2} = M_R \odot M_R$ is verified.

<u>Notation:</u> $M_R \odot M_R$ is denoted by $(M_R)^2_{\odot}$

Similarly, $\underbrace{M_R \odot M_R \ldots \odot M_R}_{'n'\text{ times}} = (M_R)^n_{\odot}$

i.e. $M_{R^n} = (M_R)^n_{\odot}, \quad n \geq 2$

Now, we know that, $R^{\infty} = \bigcup_{i=1}^{\infty} R^i$

i.e. $R^{\infty} = R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots$

we can check that If R and S are two relations then $M_{R \cup S} = M_R \vee M_S$.

if we extend this idea we have,

$M_{R^{\infty}} = M_{R^1} \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \vee \ldots$

i.e. $M_{R^{\infty}} = M_{R^1} \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot} \vee (M_R)^4_{\odot} \vee \ldots$

thus, we got another way of calculating $M_R{}^2, M_{R^3}, \ldots\ldots, M_{R^n}, \ldots, M_R{}^{\infty}$ and which in turn gives the sets $R^2, R^3, \ldots, R^n, \ldots\ldots, R^{\infty}$.

Check your progress

1. Consider the following diagraph and answer following

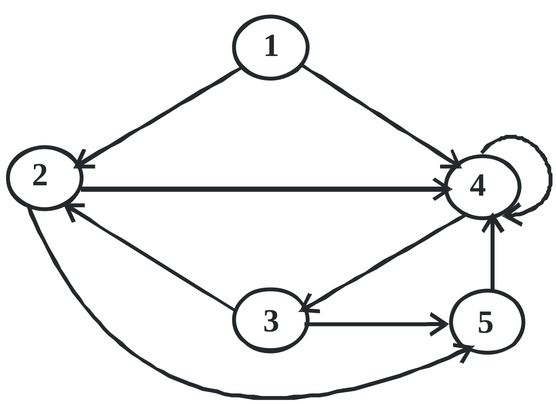


Fig. 4.9

(a) List all paths of length ‘2’

(b) List of all paths of length ‘2’ starting from ‘3’

(c) List all cycles

(d) All cycles starting at '1'

2. For example (1) find M_{R^2} and M_{R^3} ?

3. Prove that if R and S are two relations then $M_{R \cup S} = M_R \vee M_S$

4.5 PROPERTIES OF RELATIONS:

4.5.1 Reflexive and Irreflexive Relations

A relation on set 'A' is reflexive if $(x, x) \in R \;\forall\; x \in A$ [or xRx $\forall\; x \in A$]

A relation on set 'A' is irreflexive if $(x, x) \notin R \;\forall\; x \in A$ [or $x \not R x \;\forall\; x \in A$]

For e.g. (1) Let $A = \{1, 2, 3, 4\}$ with Relations R,S,T on A.

If $R = \{(1,1), (1, 2), (2, 2), (3, 3), (4, 4), (4, 2)\}$ then 'R' is reflexive.

If $S = \{(1,1), (2,1), (3,3), (4,3), (4,4)\}$ then 'S' is not reflexive. $(\because 2 \not R\, 2)$ and also 'S' is not irreflexive. $(\because (3,3) \in R \text{ or } (2,2) \in R)$. if $T = \{(1, 2), (2, 3), (3,1)\}$ then T is irreflexive.

Note:

(1) $\Delta = \{(x, x) / x \in A\}$ is called as an equality relation on 'A'.

(2) 'R' is reflexive iff $\Delta \subseteq R$.

(3) 'R' is irreflexive iff $\Delta \cap R = \emptyset$

(4) If $R = \emptyset$, an empty relation then 'R' is not reflexive since $(x,\; x) \notin R \;\forall\; x \in A$. However, R is irreflexive.

(5) Let 'R' be a reflexive relation on set 'A' then matrix of relation M_R must have diagonal elements as 1.

(6) If 'R' is irreflexive then M_R must have diagonal elements as zero's.

4.5.2 Symmetric, Asymmetric and Antisymmetric relations

(1) A relation on set 'A' is symmetric if whenever $(a,\; b) \in R$, then $(b,\; a) \in R$.

(2) A relation on set 'A' is asymmetric if whenever $(a,\; b) \in R$, then $(b,\; a) \notin R$.

(3) A relation on set A is antisymmetric if whenever

$(a, b) \in R$, & $(b, a) \in R$ then $a = b$.

For Examples:

(1) Let $A = \mathbb{N}$, 'R' be a relation on 'A' such that xRy iff 'x' divides 'y'.

(a) If xRy (i.e. 'x' divides 'y')

then yRx or $y \not{R}\, x$

($\because$ 'y' may or mayn't divide 'x')

For eg: 2 R8 (as 2|8) but $8 \not{R}\, 2 \, (\because 8 \times 2)$

$\therefore$ R isn't asymmetric

(b) If a = b = 2 then aRb as well as bRa

$\therefore$ R is not asymmetric

(c) If 'a' and 'b' are such that a/b and b/a

A / b and b / a gives a = b

$\therefore$ R is antisymmetric

(2) Let A = set of all lines in a xy-plane.

(a) If ℓ and m are in A such that ℓ R m then

m R ℓ ($\because \ell \, R \, m \Rightarrow \ell \,||\, m \Rightarrow m \,||\, \ell \Rightarrow m \, R \, \ell$)

$\therefore$ R is symmetric

(b) R is not asymmetric as $\ell \, R \, m \Rightarrow mR \, \ell$

(c) R is not antisymmetric as we can have 2 distinct lines | | to each other butn't equal.

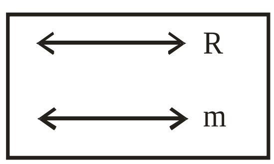


Fig. 4.10

(3) Let $A = \mathbb{N}$, 'R' is a relation such that x R y iff x < y then R is not symmetric but R is asymmetric. ($\because x \, R \, y \Rightarrow x < y$

$\Rightarrow y \not< x$

$\Rightarrow y \not{R}\, x$)

Notes:

1) The matrix M_R of a symmetric relation satisfies the property that

$M_R = M_R^t$ i.e. if $m_{ij} = 1$ then $m_{ji} = 1$ and if $m_{ij} = 0$ then $m_{ji} = 0$

2) The matrix M_R of an asymmetric relation satisfies the property if $m_{ij} = 1$ then $m_{ji} = 0$ and $m_{ii} = 0$ $\forall$ i (i.e. diagonal elements are zero)

3) Relation 'R' is antisymmetric means x R y and y R x $\Rightarrow x = y$ contrapositive of this statement is, if $x \neq y \Rightarrow x\not{R}y$ or $y\not{R}x$

i.e. M_R of antisymmetric relation satisfied the property that if $i \neq j$, then $M_{ij} = 0$ or $M_{ji} = 0$

Similarly for diagraphs we have,

4) The diagraph of symmetric relation has the property that if there an edge from i to j, then there is an edge from j to i.

5) If R is an asymmetric relation, then if there an edge from i to j so there can't be any edge from j to i and there can't be any cycle of length '1'.

6) If 'R' is an antisymmetric relation, then for different i and j there can not be an edge from vertex 'i' to vertex 'j' and an edge from vertex 'j' to vertex 'i'. (we can't say anything if $i = j$)

4.5.3 Transitive Relations

A relation 'R' on set 'A' is said to be transitive if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

[i.e. if xRy and $yRz \Rightarrow xRz$]

For Example:

(1) Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1,1), (1, 2), (2, 2), (2, 4), (1, 4)\}$ and $S = \{(3, 2), (2,1), (1, 4), (4, 2), (1, 2)\}$ then we can check 'R' is transitive but 'S' is not. [$\because (3, 2)$ & $(2,1) \in S$ but $(3,1) \notin S$]

(2) Let $A = \mathbb{N}$ & Let 'R' be a relation $\leq$. Then, if xRy and yRz i.e. $x \leq y$ & $y \leq z$

We have $x \leq y \leq z \Rightarrow xRz$

$\therefore$ R is transitive.

Notes:

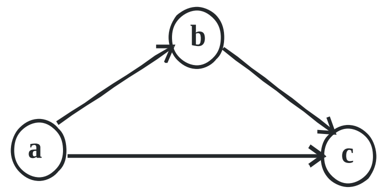
(1) A matrix M_R of relation 'R' has the property, if $m_{ij} = 1$ & $m_{jk} = 1$

Then $m_{ik} = 1$

(2) from above point (1) we can see that if $(M_R)^2_{\odot} = m_R$ then R is transitive but converse is not true.

(3) If there is a path of length '2' from 'a' to 'c' then there has to be a path of length '1' from 'a' to 'c' inorder to have 'R' transitive.

i.e. if (a, c) $\in R^2$ then $(a, c) \in R$



i.e. $R^2 \subseteq R$

$\therefore$ R is transitive iff $R^2 \subseteq R$

Fig. 4.11

(4) more generally we have,

'R' is transitive iff $R^n \subseteq R \quad \forall\ n \geq 1$

Check your progress

1. Determine whether the following relation is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.

   Let $A = \{1, 2, 3, 4\}$

   (a) $R_1 = \{(1,1), (2, 2)\}$

   (b) $R_2 = \{(1,1), (2, 3), (3, 2), (2, 2)\}$

   (c) $R_3 = \{(2, 3), (3, 2), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

   (d) $R_4 = \{(2, 3), (3, 2), (1, 4), (4, 2), (1, 2)\}$

2. Determine whether the relation 'R' on the set $A = \{1, 2, 3\}$ whose matrix M_R and Diagraph are given is reflexive, irreflexive, symmetric, asymmetric, antisymmertic or transitive.

   (a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b)

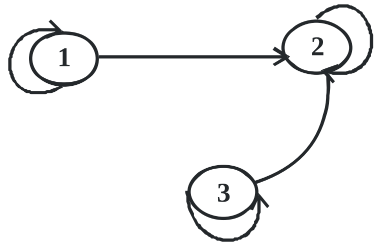


Fig. 4.12

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(d)

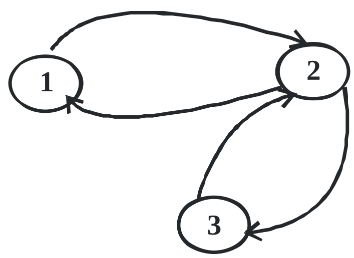


Fig. 4.13

4.6 LET US SUM UP

We started the definition of product sets which is useful in defining relation from one set to other. Then we saw different ways of representing a relation. Which is useful in understanding the concepts in more better way. Then we saw very important definition of a path in a relation and then concepts of paths of different length and then finally matrix version of it. At the end we saw different properties of a relation which is useful in coming chapters.

4.7 REFERENCES FOR FURTHER READING:

(1) Discrete structures by B. Kolman HC Busloy, S Ross PHI Pvt. Ltd.

(2) Discrete mathematics and its application, Keneth H. Rosen TMG.

(3) Discrete structures by Liu.

4.8 UNIT END EXERCISES:

1. Find R^{∞} for the relation 'R' whose diagraph is

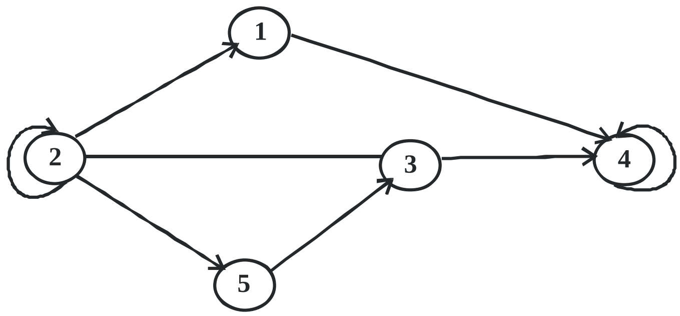


Fig. 4.14

Q.2 Calculate M_{R^4} for a relation $R = \{(1,1), (3,2), (1,4), (2,4)\}$ on set $A = \{1, 2, 3, 4\}$

Q.3 Determine whether following relations are reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.

(a) $A = \mathbb{Z}$, x R y iff x + y is an even number.

(b) $A = \mathbb{R}$, x R y iff $x^2 + y^2 = 9$

(c) $A = \mathbb{N}$, x R y iff $x \le y$

(d) $A = \mathbb{Z}$, x R y iff $(x - y) \le 3$

Q.4 Define a relation on $A = \{1, 2, 3, 4\}$ that is

(a) reflexive butn't symmetric

(b) transitive butn't reflexive

(c) antisymmetric and reflexive

(d) irreflexive and transitive

Q.5 Prove that if 'R' is symmetric then R^2 is also symmetric.

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